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PATCHWORKING SINGULAR ALGEBRAIC CURVES II

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ABSTRACT

In this paper we present various particular versions of the general patchworking procedure for the construction of reduced algebraic curves with prescribed singularities on algebraic surfaces. Among the main examples are a deformation of reducible algebraic curves on reducible algebraic surfaces in the presence of non-transverse intersections of a curve with the singular locus of a surface, and a deformation of curves with multiple components. As an application we deduce a significant asymptotical improvement for the sufficient existence criterion of algebraic curves with arbitrary prescribed singularities in given linear systems on smooth projective algebraic surfaces.

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1. Introduction

The patchworking construction, based on the Viro "gluing" method [22, 23, 24, 25, 26, 27], which was invented in 1979–80 for the construction of real algebraic non-singular hypersurfaces with prescribed topology, and which provided a major breakthrough in Hilbert's 16th problem [9], was later modified for the construction of algebraic curves with many prescribed singularities in a plane [15, 17] or other algebraic surfaces [12, 19], and, more generally, hypersurfaces with prescribed singularities in smooth algebraic varieties [18], construction of polynomials with prescribed critical points [16, 17], vector fields with many limit cycles and prescribed singularities [10], and some other problems, for example, enumeration of singular curves [14, 20]. We should also like to mention that the patchworking construction appears to be useful in the symplectic setting as well (cf. [4, 11]).

In [21] we proved a general patchworking theorem for the construction of curves with prescribed singularities in a given linear system on an irreducible algebraic surface, which covers and strengthens almost all previously known procedures. Namely, one includes the given algebraic surface into a one-dimensional flat family $X \to (\mathbb{C}, 0)$ which degenerates into a reduced, reducible surface X_0, then chooses a singular (reducible) curve C_0, belonging to the limit linear system in X_0, and, finally, deforms C_0 into a curve C_t in a general fiber X_t, keeping the prescribed singularities of the initial curve C_0.

In the present paper we elaborate in detail two particular versions of the general patchworking procedure.

The version, considered in section 2, deals with the case of a curve $C_0 \subset \Sigma_0$ having non-transverse intersections with the singular locus $\mathrm{Sing}(X_0)$ of X_0. We point out that in preceding patchworking procedures, the components of C_0 were required to be non-singular along $\mathrm{Sing}(X_0)$ and to intersect it transversally (see [17, 18]). The reason was the absence of tools to control the behavior of C_0 in the non-transverse case. In [1, 2, 14] one can find a technically tricky statement on the deformation of a point, where two components of C_0 are non-singular and tangent to a non-singular line of $\mathrm{Sing}(X_0)$, and the idea is based on the fact that the singularity of C_0 is planar (just A_k), and one can thoroughly study the versal deformation of such a singularity. The statement itself was used in the construction of nodal curves of surfaces of general type [2, 3], and later appeared in the tropical approach to enumerative geometry [20]. Here we suggest a completely different idea which allows us to treat a large class of singularities, not necessarily planar. Namely, we blow up the three-fold X at the singular point so that X_0 obtains exceptional divisors as new components, then we add to C_0 the appropriate new components situated in the exceptional divisors (we call them **admissible patchworking patterns**), and thus reduce the problem to the case of a non-singular transverse intersection.

Another version of the patchworking construction (see section 3) describes the deformations of non-reduced curves in smooth algebraic surfaces. To obtain controllable deformations of a multiple component of a given curve we take the trivial family of surfaces and blow it up along that multiple component. The central surface of the new family contains a geometrically ruled surface as a component, in which we insert a suitable patchworking pattern. The class of deformations, which can be produced in this way, is basically parametrized by curves in some linear system on the ruled component.

At last, in section 4, we demonstrate an application of our patchworking procedures, which results in a new sufficient numerical existence condition for irreducible curves with arbitrary prescribed singularities in given linear systems on smooth algebraic surfaces. Asymptotically, the new condition significantly improves the previously known general conditions [12, 19], and gets rid of non-numerical conditions, which have been rather hard to verify.

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2. Deformation of curves with non-transversal boundary conditions

In this section we extend the geometric patchworking procedure to the case of a non-transversal intersection of components of C_0. We use the notation of Section 2.2 [21].

Consider a pattern consisting of:

- a one parameter flat family of projective surfaces $\pi\colon X \to T$ over a smooth base T,
- a family of invertible sheaves $\mathcal{L}_t$ on $X_t = \pi^{-1}(t)$, i.e. an invertible sheaf $\mathcal{L}$ on X (up to a twist by $\pi^*\mathcal{F}$ where $\mathcal{F}$ is a line bundle on T), and
- a section $\xi_0 \in H^0(X_0, \mathcal{L}_0)$, whose set of zeroes is denoted by $C_0 = \bigcup_{i=1}^k C_0^i$.

In [21] we developed a patchworking procedure which, under appropriate assumptions, provided us with a flat deformation C_t of C_0, preserving the singularities. Namely, we assumed that:

$X1$. X_t is reduced and irreducible for any $t \neq 0$, where $0 \in T$ is a distinguished point.

$X2$. $X_0 = \bigcup_{i=1}^{k} \Sigma^i$ is a union of reduced and irreducible surfaces such that $\dim(\Sigma^i \cap \Sigma^j \cap \Sigma^k) = 0$, for any three distinct indices i, j, k.

$S1$. C_0 has only isolated singular points and all these points are smooth points of X. If $p \in \mathrm{Sing}(X_0) \cap \mathrm{Sing}(C_0)$ then $p \in \Sigma^i \cap \Sigma^j$ for some i, j.

$S2$. C_0^i are reduced.

$S3$. $C_0 \cap \Sigma^i \cap \Sigma^j$ is reduced for any $i \neq j$.

$S4$. $C_0 \cap \Sigma^i \cap \Sigma^j \cap \Sigma^k = \emptyset$, for any three distinct indices i, j, k.

$S5$. For any $p \in C_0 \cap \Sigma^i \cap \Sigma^j$, there exists an open analytic neighborhood $p \in U \subset X$ such that $X_0 \cap U \subset U$ is a **quasi-normal crossing divisor**; i.e. either it is a normal crossing divisor or the pair $U \to U_\epsilon(0) \subset T$ is isomorphic to an open analytic neighborhood of 0 of the pair $\mathrm{Spec}\,\mathbb{C}[x, y, z, t]/(xy - t^l) \to \mathrm{Spec}\,\mathbb{C}[t]$ for some positive integer l.

Notation 2.1: (1) Let $n(i)$ be the number of the singular points of C_0^i. We denote these singular points by $p_1^i, \ldots, p_{n(i)}^i$ and their singularity types by $\mathcal{S}_j^i = \mathcal{S}(p_j^i)$, $j = 1, \ldots, n(i)$. The types $\mathcal{S}_j^i$ can coincide for different j.

(2) Let $\mathcal{I}_{p_j^i} \subset \mathcal{O}_{X_0, p_j^i}$ be the equisingular/equianalytic ideal of the singular point p_j^i. We denote by $\mathcal{I}^{es/ea}$ the equisingular/equianalytic ideal sheaf of the zero-dimensional scheme concentrated at $\bigcup_{i,j} p_j^i \subset X_0$, which is defined locally at p_j^i by the ideal $\mathcal{I}_{p_j^i}$.

Under these assumptions and notation we proved

THEOREM 2.2 (Weak Patchworking Theorem): *Assume that*

$$H^1(X_0, \mathcal{I}^{es/ea} \otimes \mathcal{L}_0) = 0. \tag{1}$$

Then there exists some open neighborhood $U_\epsilon = U_\epsilon(0) \subset T$ and a family of curves $C_t \in |\mathcal{L}_t|$, $t \in U_\epsilon$, having $\sum_i n(i)$ singular points of types $\{\mathcal{S}_j^i\}_{1 \le j \le n(i)}^{1 \le i \le k}$, respectively, as their only singularities.

Some generalization (Theorem 2.15 [21]) of the last theorem was proven as well.

It turns out that in many cases the condition $S3$ is not satisfied; see, for example, [2]. However, one can still prove a version of the patchworking theorem, which is the goal of the current section. It is clear that even in the case when $S3$ is false, we can carry out the same "gluing" procedure as before with the only difference being that the points of intersection of the components of C_0 give rise to some new singularities. So the problem is to control the types and the number of "new" singular points which will appear.

The only known approach to this type of problem is a straightforward computation. This approach was successfully applied in the nodal case (cf. [1]). However, it seems to be impossible to extend this type of computation to the case of more complicated singularities.

Our approach is different. We suggest reducing the problem, using geometric patchworking, to the problem of describing singular curves in toric surfaces. To do this, one has to blow up the family X along an appropriate non-reduced subscheme, concentrated at the "bad" intersection points, in such a way that the proper transforms $\widetilde{C_0^i}$ and $\widetilde{\Sigma^i}\cap(\widetilde{\Sigma^j}\cup E)$ will intersect transversally (the same when switching i and j).

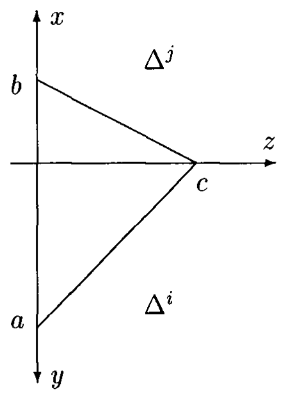

Figure 1. Blow up of a boundary singularity.

Let us explain our idea in more detail for the case of semiquasihomogeneous singularities of C_0^i and C_0^j at $p \in C_0^i \cap C_0^j$. Denote a local parameter on T by t. We know that in some neighbourhood of p, X is given by $xy = t^k$. Let z be a third local coordinate. Assume that the Newton diagrams of C_0^i and C_0^j at p are Δ^i and Δ^j as in Figure 1. Denote $n = \mathrm{lcm}(a, b, c)$, and consider the base change $\widetilde{X} = X \times_T S$ where σ: $S \to T$ is a totally ramified (at $0 \in T$) $n(1/a + 1/b)$-covering of T. Denote the local parameter on S by s (we can assume that $s^{n(1/a+1/b)} = \sigma^*(t)$). Now we can consider the zero-dimensional subscheme Z concentrated at $\widetilde{p} \in \widetilde{X}$ which is defined by the ideal $\mathcal{I}$ with $\mathcal{I}_{\widetilde{p}} \subset \mathcal{O}_{\widetilde{X},\widetilde{p}}$ generated by

$$\{x^j z^i s^{n(1-j/b-i/c)}\}_{0\le i,j\in\mathbb{Z}}^{i/c+j/b\le 1} \bigcup \{y^j z^i s^{n(1-j/a-i/c)}\}_{0\le i,j\in\mathbb{Z}}^{i/c+j/a\le 1}. \tag{2}$$

Consider the new flat projective family of surfaces $Y = Bl_Z(\widetilde{X}) \to S$.

Remark 2.3: The generic fiber of the new family Y is isomorphic to the generic fiber of X and $Y_0 = Bl(X_0) \cup E$.

Notation 2.4: We introduce the following notation for the projections.

- $\pi_{Y\widetilde{X}}: Y \to \widetilde{X}$,
- $\pi_{\widetilde{X}X}: \widetilde{X} \to X$,
- $\pi_{YX} = \pi_{\widetilde{X}X} \circ \pi_{Y\widetilde{X}}$,
- $\pi: Y \to S$.

Finally, we consider the new family of sheaves $\mathcal{E} = \pi^*_{YX}(\mathcal{L}) \otimes \pi^*_{Y\widetilde{X}}\mathcal{I}$ and a new section

$$\zeta = \pi^*_{YX}(\xi_0) \uplus \zeta_E \in H^0(Y_0, \mathcal{E}_{|Y_0}),$$

where $\zeta_E = \zeta_{|E} \in \mathcal{E}_{|Y_0}(E)$ is any (admissible) section, cf. Definition 2.7, defining a curve with isolated singular points as its only singularities. The latter curve $\{\zeta_E = 0\} = C_E \subset E$ we call **an admissible patchworking pattern**.

CLAIM 2.5:

- *The exceptional divisor E is the toric surface associated with the triangle $\{(0,b),(0,-a),(c,0)\} \subset \mathbb{Z}^2 \otimes \mathbb{R}$;*
- *the intersection $\pi^{-1}_{YX}(C^i_0) \cap (\widetilde{\Sigma^i} \cap E)$ is transversal;*
- *$\pi^{-1}_{YX}(C^i_0) \cap \widetilde{\Sigma^i} \cap E \cap \widetilde{\Sigma^j} = \emptyset$;*
- *in the neighborhood of any $p \in \pi^{-1}_{YX}(C^i_0) \cap \widetilde{\Sigma^i} \cap E \subset X$ the divisor $\widetilde{\Sigma^i} \cup E \subset X$ is a quasi-normal crossing divisor.*

Proof: The proof of the claim is a straightforward computation and we omit it here. ∎

COROLLARY 2.6: (1) *The patchworking pattern $(Y, \mathcal{E}, \zeta)$ satisfies conditions $X1$, $X2$, $S1$, $S2$, $S3$, $S4$, $S5$.*

(2) *If condition (1) of Theorem 2.2 is satisfied for the new data $Y, \mathcal{E}, \zeta$ then for a generic $t \in T$ there exists a curve C_t (defined by a section $\xi_t \in H^0(X_t, \mathcal{L}_t)$) having $\{S^i_j\}^{1\le i\le k}_{1\le j\le n(i)} \cup \mathrm{Sing}(\zeta_E)$ as its only singularities.*

Now we would like to present a generalization of Theorem 2.2 and of the last corollary. To do this we need the following

Definition 2.7: (1) In the above notations, let $p \in \Sigma^i \cap \Sigma^j$ be a common point of the curves C^i_0 and C^j_0, which is a semiquasihomogeneous singular point for both the curves. We define the sheaves $\mathcal{J}^{ip}$ on Σ^i and $\mathcal{J}^{jp}$ on Σ^j to be the ideal sheaves of the zero-dimensional schemes concentrated at p with stalks at p given by $(\mathcal{J}^{ip})_p = \text{Ideal}(\Delta^i)$ and $(\mathcal{J}^{jp})_p = \text{Ideal}(\Delta^j)$, where Δ^i, Δ^j are the (local) Newton diagrams of the singular point p of C_0^i, C_0^j, respectively (see Figure 1), and Ideal$(*)$ is defined by vanishing of the coefficients under the Newton diagram.

(2) A set $A^p_{X,\mathcal{L},\xi_0}$ of singularity types is called admissible if there exists a section $\zeta_E \in H^0(E, \mathcal{E}_{|_E})$, called admissible section, satisfying

- $\zeta = \pi^*_{YX}(\xi_0) \uplus \zeta_E$ is a section of $\mathcal{E}_{|_{Y_0}}$ (denote by C_0 the curve defined by ζ),
- $C_0 \cap \text{Sing}(E) = \emptyset$,
- $C_0 \cap \Sigma^i \cap E$ and $C_0 \cap \Sigma^j \cap E$ are reduced,
- $C_0 \cap \Sigma^i \cap \Sigma^j \cap E = \emptyset$,
- $C_0 \cap E$ has exactly $|A^p_{X,\mathcal{L},\xi_0}|$ singular points of types $A^p_{X,\mathcal{L},\xi_0}$, and
- for the equisingular (equianalytic) ideal $\mathcal{I}_{A^p_{X,\mathcal{L},\xi_0}}$ of $\text{Sing}(C_0 \cap E)$,

$$H^1(E, \mathcal{I}_{A^p_{X,\mathcal{L},\xi_0}} \otimes \mathcal{O}_E(C_0 \cap E - E \cap (\widetilde{\Sigma}^i \cup \widetilde{\Sigma}^j))) = 0. \tag{3}$$

THEOREM 2.8: *Assume that we are given a flat projective family of surfaces $X \to T$ over a smooth base T and an invertible sheaf $\mathcal{L}$ on X satisfying $X1$ and $X2$. Consider a section $\xi_0 \in H^0(X_0, \mathcal{L})$ and its zero set $C_0 = \bigcup C_0^i$ satisfying $S1, S2, S4, S5$ and assume that, for any $i \neq j, p \in C_0 \cap \Sigma^i \cap \Sigma^j$, either $C_0 \cap \Sigma^i \cap \Sigma^j$ is reduced at p, or C_0^i and C_0^j have semiquasihomogeneous singularities at p. As before we denote the singularity types of singular points of C_0 in $X_0 - \bigcup_{\alpha\neq\beta}(\Sigma^\alpha \cap \Sigma^\beta)$ by S^i_j. The singular points of C_0 in $\Sigma^i \cap \Sigma^j$ which are not ordinary nodes we denote $p_{ij\alpha} \in \Sigma^i \cap \Sigma^j$. Consider the equisingular/equianalytic ideal $\mathcal{I}^{es/ea}$ of the "internal" singular points of C_0 (see Notation 2.1), and, for any boundary singular point $p_{ij\alpha}$, define the sheaves $\mathcal{J}^{i,p_{ij\alpha}}$ and $\mathcal{J}^{j,p_{ij\alpha}}$ as in Definition 2.7. Assume that*

$$H^1\left(X_0, \bigotimes_{\alpha,j,i}(\mathcal{J}^{i,p_{ij\alpha}} \otimes \mathcal{J}^{j,p_{ij\alpha}}) \otimes \mathcal{I}^{es/ea} \otimes \mathcal{L}_0\right) = 0. \tag{4}$$

Then, for any collection of admissible sets $A^{p_{ij\alpha}}_{X,\mathcal{L},\xi_0}$ of singularity types, there exists an open neighborhood $U_\epsilon = U_\epsilon(0) \subset T$ and a family of sections $C_t \in H^0(X_t, \mathcal{L}_t)$, $t \in U_\epsilon$, having $\{S^i_j\}^{1\leq i\leq k}_{1\leq j\leq n(i)} \cup A_{X,\mathcal{L},\xi_0}$ as its only singularities, where $A_{X,\mathcal{L},\xi_0} = \bigcup A^{p_{ij\alpha}}_{X,\mathcal{L},\xi_0}$.

Proof: As before we start with an appropriate base change and get family $\widetilde{X} \to S$. Then we blow up this family along the zero-dimensional scheme concentrated at $\bigcup p_{ij\alpha}$, and given by sheaf of ideals $\mathcal{I}$ with stalks $\mathcal{I}_{p_{ij\alpha}}$ defined similarly to (2). We denote the new family by $Y \to S$.

Next we define the new family of line bundles by $\mathcal{E} = \pi_{YX}^*\mathcal{L} \otimes \pi_{YX}^*\mathcal{I}$. Now, given any collection of admissible singularity types $A_{X,\mathcal{L},\xi_0}$, by the definition we can construct a section ζ of $\mathcal{E}_0$, such that $Z(\zeta)$ has $\{\mathcal{S}_j^i\}_{1\le j\le n(i)}^{1\le i\le k} \cup A_{X,\mathcal{L},\xi_0}$ as its only singularity outside of the intersection curves of several components of Y_0. Moreover, $Z(\zeta)$ intersects transversally any of these curves.

It is clear that the triple $(Y,\mathcal{E},\zeta)$ satisfies conditions $X1, X2, S1-S5$. So, if condition (1) is also satisfied, then we can use Theorem 2.2 to derive the result. It remains to prove that

$$H^1(Y_0, \mathcal{I}_\zeta^{es/ea} \otimes \mathcal{E}_0) = 0. \tag{5}$$

Consider the exact sequence

$$\bigoplus_E H^1(E, \mathcal{I}_{\zeta_{|E}}^{es/ea} \otimes \mathcal{O}_E(C_0 \cap E - E \cap (\widetilde{\Sigma}^i \cup \widetilde{\Sigma}^j))) \to$$

$$\to H^1(Y_0, \mathcal{I}_\zeta^{es/ea} \otimes \mathcal{E}_0) \to H^1(\widetilde{X_0}, \mathcal{I}_{\zeta_{|\widetilde{X_0}}}^{es/ea} \otimes \mathcal{E}_0).$$

The first group is zero by the definition of the admissible singularity classes. The last group is zero, since

$$H^1(\widetilde{X_0}, \mathcal{I}_{\zeta_{|\widetilde{X_0}}}^{es/ea} \otimes \mathcal{E}_0) = H^1\left(X_0, \bigotimes_{\alpha,j,i} (\mathcal{J}^{i,p_{ij\alpha}} \otimes \mathcal{J}^{j,p_{ij\alpha}}) \otimes \mathcal{I}_{\xi_0}^{es/ea} \otimes \mathcal{L}_0\right) = 0.$$

Hence (5) is satisfied, and we are done. ∎

To illustrate the theorem we consider the following example:

Example 2.9: Let $\Delta \subset \mathbb{R}^2$ be a convex non-degenerate lattice polygon, $\mathrm{Tor}(\Delta)$ the toric surface associated with Δ. The monomials x^iy^j, $(i,j) \in \Delta \cap \mathbb{Z}^2$, generate a very ample linear system $\Lambda(\Delta) = |\mathcal{L}(\Delta)|$ on $\mathrm{Tor}(\Delta)$. We claim that there is a rational nodal curve $C \in \Lambda(\Delta)$, which, for any edge $\sigma \subset \Delta$, meets $\mathrm{Tor}(\sigma)$ at a unique point, and, furthermore, C is non-singular at all these points. In other words, C crosses any $\mathrm{Tor}(\sigma)$, $\sigma \subset \partial\Delta$, at one non-singular point with multiplicity equal to $|\sigma|$, the lattice length of the edge σ. Moreover, for any two fixed distinct edges $\sigma_1, \sigma_2 \subset \partial\Delta$ and any two fixed generic points $z_1 \in Tor(\sigma_1), z_2 \in Tor(\sigma_2)$, we can find C as above passing through z_1, z_2 and the family of such curves has expected dimension.

It is not difficult to show that such a rational curve does exist, however it is not evident that one can find a *nodal* rational curve. For example, a relatively simple case of a triangular Δ requires some computation (see [20], Lemma 3.5). Our patchworking theorem allows one to extend the required statement from the case of a triangle to an arbitrary lattice polygon Δ. Namely, we prove the statement by induction on the number of vertices in Δ.

The base of induction is the case of a triangle, and it is done in [20], Lemma 3.5. To perform the induction step, we start with Δ which is not a triangle. Let $v_0, v_1, \dots, v_{n+1}$, $n \geq 2$, be the vertices of Δ. We divide Δ by the diagonal (v_0, v_2) into a union

$$\Delta_1 \cup \Delta_2 = \text{conv}\{v_0, v_1, v_2\} \cup \text{conv}\{v_2, \dots, v_{n+1}, v_0\},$$

and work in the setting of Example 2.2 [21]. We will assume that $z_i \in \text{Tor}(\partial\Delta_i)$.

For $i = 1, 2$, we take a rational nodal curve $C_0^i \in \Lambda(\Delta_i)$, which meets any divisor $\text{Tor}(\sigma)$, $\sigma \subset \partial\Delta_i$, at a non-singular point with multiplicity $|\sigma|$. Such curves do exist by the induction assumption, and moreover, they can be chosen so that, for the common edge $\sigma = \Delta_1 \cap \Delta_2$, it holds that $C_0^1 \cap \text{Tor}(\sigma) = C_0^2 \cap \text{Tor}(\sigma) = \{z\}$ is a generic point on $\text{Tor}(\sigma)$. Hence, we have constructed a patchworking pattern, which satisfies the required conditions with the only exception being condition $S3$.

By [20], Lemma 5.5 (i), the set of $|\sigma| - 1$ nodes is admissible on $\text{Tor}(E)$, where E is the triangle with vertices $(0,1), (0,-1), (|\sigma|, 0)$. So, if condition (4) is satisfied for the presented patchworking pattern, then we can apply Theorem 2.8 to derive the existence of an irreducible reduced curve $C_t \in \Lambda(\Delta)$ having $|\,\text{Int}(\Delta) \cap \mathbb{Z}^2|$ nodes as its only singularities. Since the arithmetic genus of any curve in $\Lambda(\Delta)$ is $|\,\text{Int}(\Delta) \cap \mathbb{Z}^2|$, the resulting curve is rational. To preserve the tangency conditions to the divisors at infinity, we should modify condition (4) in a sense of Remark 3.2 (2) [21]. Namely, instead of the ideal

$$\mathcal{I} = \bigotimes_{\alpha,j,i} (\mathcal{J}^{i,p_{ij\alpha}} \otimes \mathcal{J}^{j,p_{ij\alpha}}) \otimes \mathcal{I}^{es/ea}$$

we consider $\mathcal{I} \otimes \mathcal{J}$, where $\mathcal{J}_{|_{Tor(\Delta_i)}}$ is the ideal concentrated at the intersection points of C_0^i with the lines L, corresponding to the common edges of Δ_i and Δ, where it is locally given by $\{f \in \mathcal{O}_z | (f \cdot L)_z \geq (C_0^i \cdot L)_z - 1\}$ if $z \neq z_1, z_2$, and $\{f \in \mathcal{O}_z | (f \cdot L)_z \geq (C_0^i \cdot L)_z - 1\}$ if $z = z_1$ or $z = z_2$.

So, it remains to check modified condition (4). Denote $\mathcal{F} = \mathcal{I} \otimes \mathcal{J} \otimes \mathcal{L}_0$, $\mathcal{F}_1 = \mathcal{F}_{|_{\text{Tor}(\Delta_1)}}$, $\mathcal{F}_2 = \mathcal{F}_{|_{\text{Tor}(\Delta_2)}}$, $\mathcal{F}_{12} = \mathcal{F}_{|_{\text{Tor}(\Delta_1 \cap \Delta_2)}}$ and consider the exact sequence

$$0 \to \mathcal{F} \to \mathcal{F}_1 \oplus \mathcal{F}_2 \to \mathcal{F}_{12} \to 0.$$

Due to the induction assumption, $H^1(\text{Tor}(\Delta_i), \mathcal{F}_i) = 0$, hence, in order to check modified condition (4), i.e. to prove $H^1(\cup \text{Tor}(\Delta_i), \mathcal{F}) = 0$, it is sufficient to show that the map

$$(6) \qquad H^0(\mathrm{Tor}(\Delta_1), \mathcal{F}_1) \oplus H^0(\mathrm{Tor}(\Delta_2), \mathcal{F}_2) \to H^0(\mathrm{Tor}(\Delta_1) \cap \mathrm{Tor}(\Delta_2), \mathcal{F}_{12})$$

is surjective. But the map $H^0(\mathrm{Tor}(\Delta_1), \mathcal{F}_1) \to H^0(\mathrm{Tor}(\Delta_1) \cap \mathrm{Tor}(\Delta_2), \mathcal{F}_{12})$ is surjective (one can easily derive this from the Riemann–Roch Theorem), therefore (6) holds, and we are done.

3. Deformation of curves with multiple components

Assume that we are given a plane curve $C \subset \mathbb{P}^2$ of the form $C = C_1 + kC_2$, $k > 1$, where C_1 and C_2 are smooth curves of degrees d_1, d_2, intersecting transversally. One would like to describe the deformation space of the singularity of C. This space is too big, but one can describe some natural small subspace of it. Namely, consider $\mathbb{P}^2 \times \mathbb{P}^1$ and define X to be the blow up of $\mathbb{P}^2 \times \mathbb{P}^1$ along $C_2 \times 0 \subset \mathbb{P}^2 \times \mathbb{P}^1$. Then X admits natural projections $\pi: X \to \mathbb{P}^1$ and $\sigma: X \to \mathbb{P}^2$. If $t \neq 0$ then $\pi^{-1}(t) = \mathbb{P}^2$ and if $t = 0$ then $\pi^{-1}(0) = \mathbb{P}^2_0 \cup E$, where the exceptional divisor $E = \mathbb{P}roj(\mathcal{O}_{C_2} \oplus \mathcal{O}_{C_2}(C_2))$ is a ruled surface over C_2.

Now we consider the family $\mathcal{L} = \sigma^* \mathcal{O}_{\mathbb{P}^2}(d_1 + kd_2) \otimes \mathcal{O}_X(-kE)$ of line bundles on X. Thus

- $\mathcal{L}_{|X_t} = \mathcal{O}_{\mathbb{P}^2}(d_1 + kd_2)$ for $t \neq 0$,
- $\mathcal{L}_{|\mathbb{P}^2_0} = \mathcal{O}_{\mathbb{P}^2}(d_1)$,
- $\mathcal{L}_{|E} = \mathcal{O}_E(kC_2 + (d_1 d_2 + kd_2^2)F)$ where F is the class of the fiber in E.

Let $\xi \in H^0(X_0, \mathcal{L}_0)$ be a section, and let $\widetilde{C} = \widetilde{C}_1 \cup \widetilde{C}_2 \subset \mathbb{P}^2_0 \cup E$ denote the curve defined by the section ξ. Assume that $\widetilde{C}_1 = C_1$. Assume also that $\widetilde{C}_2$ has only isolated singularities of some types $\mathcal{S}_1, \ldots, \mathcal{S}_r$. We denote the equisingular (equianalytic) ideal of $\bigcup \mathcal{S}_i \subset X_0$ by $I^{es/ea}_{\widetilde{C}}$.

THEOREM 3.1: *Assume that*

$$(7) \qquad H^1(E, (I^{es/ea}_{\widetilde{C}} \otimes \mathcal{L})_{|E}) = 0.$$

Then there exists a deformation $C_t \subset \mathbb{P}^2$ of $C_0 = C$ and, for any $t \neq 0$, the curve C_t has exactly r singular points of types $\mathcal{S}_1, \ldots, \mathcal{S}_r$ as its only singularities.

Proof: This theorem is an easy consequence of the geometric patchworking. First of all we have to show that

$$(8) \qquad H^1(X_0, I^{es/ea}_{\widetilde{C}} \otimes \mathcal{L}) = 0.$$

Consider the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2}(d_1 - d_2) \to I^{es/ea}_{\widetilde{C}} \otimes \mathcal{L} \to (I^{es/ea}_{\widetilde{C}} \otimes \mathcal{L})_{|E} \to 0.$$

$H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d_1 - d_2)) = 0$, thus (7) implies (8). Now one can apply Theorem 2.2 to derive the result. ∎

4. Existence of curves with prescribed singularities on algebraic surfaces

In this section we consider the problem of the existence of curves with prescribed singularities on algebraic surfaces. Namely, let Σ be a smooth projective algebraic surface, $\mathcal{L} \in Pic(\Sigma)$, and let $\mathcal{S}_1, \ldots, \mathcal{S}_r$ be singularity types. Consider the variety $V_{|\mathcal{L}|}(\mathcal{S}_1, \ldots, \mathcal{S}_r)$ parameterizing curves $C \in |\mathcal{L}|$ having exactly r singular points of types $\mathcal{S}_1, \ldots, \mathcal{S}_r$ as its only singularities. We ask the following questions:

- Is $V_{|\mathcal{L}|}(\mathcal{S}_1, \ldots, \mathcal{S}_r) \neq \emptyset$?
- Does $V_{|\mathcal{L}|}(\mathcal{S}_1, \ldots, \mathcal{S}_r)$ contain a regular component, i.e. a component of expected dimension?

A complete answer (i.e. a necessary and sufficient numerical condition on $\mathcal{L}$ and singularity types) to these questions is known only for nodal curves on very specific surfaces. However, a sufficient condition to this problem was proposed in [7, 19] in the plane case, and in [12] for general surfaces. The goal of this section is to significantly improve the numerical conditions and to get rid of some non-numerical condition given in [12], which are hard to verify. Our method is based on the patchworking procedure developed in [21]. Namely, first we prove the result for geometrically ruled surfaces. Then we degenerate the surface Σ into a union of Σ and some geometrically ruled surface, and use geometric patchworking to derive the general result. In fact this method is closely related to the deformations of curves with multiple components, presented in the previous section. Namely, the resulting singular curve is obtained as a deformation of a curve with multiple components in Σ, and the patchworking procedure allows us to describe this deformation.

4.1 Geometrically ruled surfaces. Let L be a smooth, connected, projective curve of genus g_L, and let $D \in Div(L)$ be a positive divisor. Consider the geometrically ruled surface

$$\pi_{L,D}: \Sigma_{L,D} = \mathbb{P}\mathrm{roj}(\mathcal{O}_L \oplus \mathcal{O}_L(D)) \to L.$$

It is well known that $\pi_{L,D}$ admits a unique section whose image has a negative self-intersection. We identify the image of this section with L. It is also known that $\dim NS(\Sigma_{L,D}) = 2$; moreover, $NS(\Sigma_{L,D})$ is generated by the class of a fiber $F = \pi^{-1}_{L,D}(pt)$ and the class of L, and any effective divisor $M \in Div(\Sigma_{L,D})$ is numerically equivalent to a combination of F and L with non-negative coefficients. It is important to mention that the intersection form is given by the equalities $L^2 = -deg(D)$, $F^2=0$, and $L.F = 1$.

THEOREM 4.1: *Let $S_1, \ldots, S_r$ be singularity types, let a and b be positive integers, and let $M \in Div(\Sigma_{L,D})$ be any divisor numerically equivalent to $aL + bF$. Define $m_i = s(S_i) + 1$. Without loss of generality, $m_1 \geq m_2 \geq \cdots \geq m_r$. Assume that*

$$b - 2g_L + 1 + deg(D) \geq (a+2)deg(D) > (m_1+1)deg(D), \tag{9}$$

$$(M - K_{\Sigma_{L,D}})^2 \geq \frac{deg(D)+1}{deg(D)}\left(2m_1 + 3 + \sum_{i=1}^{r}(m_i+1)^2\right), \tag{10}$$

$$(M - K_{\Sigma_{L,D}} - F)^2 \geq \frac{deg(D)+1}{deg(D)}\sum_{i=1}^{r}(m_i+1)^2, \tag{11}$$

$$(M - K_{\Sigma_{L,D}} - L)^2 \geq \frac{deg(D)+1}{deg(D)}\sum_{i=1}^{r}(m_i+1)^2. \tag{12}$$

Then, for almost every section $\alpha \in H^0(L, \mathcal{O}_L(M))$, there exists an irreducible algebraic curve $C \in |\mathcal{O}_{\Sigma_{L,D}}(M)|$ such that

(i) *C has exactly r singular points of types $S_1, \ldots, S_r$ as its only singularities,*

(ii) *$C \cap L$ is given by α (in particular C intersects L transversally),*

(iii) *the equisingular/equianalytic stratum of C is smooth at C and has the expected dimension.*

Remark 4.2: (1) Compared to the previously known existence results, cf. [12], we mention that the statement above improves significantly the leading coefficient, namely in [12] the leading coefficient was 2, which is usually much weaker than $(deg(D)+1)/deg(D)$, while the optimal coefficient is one.

(2) It is well known that $K_{\Sigma_{L,D}} \equiv_{num} -2L + (2g_L - 2 - deg(D))F$. Hence the conditions (10), (11), (12) can be reformulated as

$$(a+2)(2b - 4g_L + 4 - a\cdot deg(D)) \geq \frac{deg(D)+1}{deg(D)}\left(2m_1 + 3 + \sum_{i=1}^{r}(m_i+1)^2\right),$$

$$(a+2)(2b - 4g_L + 2 - a\cdot deg(D)) \geq \frac{deg(D)+1}{deg(D)}\sum_{i=1}^{r}(m_i+1)^2,$$

$$(a+1)(2b - 4g_L + 4 - a\cdot deg(D) + deg(D)) \geq \frac{deg(D)+1}{deg(D)}\sum_{i=1}^{r}(m_i+1)^2.$$

The following lemmas will be useful:

LEMMA 4.3: *Let $n_1, \dots, n_r \in \mathbb{N}$, and $R \in Div(\Sigma_{L,D})$ be a divisor satisfying the following three conditions:*

$$(R - K_{\Sigma_{L,D}})^2 \geq \frac{deg(D)+1}{deg(D)} \sum_{i=1}^{r} (n_i+1)^2, \tag{13}$$

$$(R - K_{\Sigma_{L,D}}).F > \max\{n_i | i = 1, \dots, r\}, \tag{14}$$

$$R - K_{\Sigma_{L,D}} \text{ is nef.} \tag{15}$$

Then for $z_1, \dots, z_r \in \Sigma$ in general position and $\nu > 0$

$$H^\nu\left(\widetilde{\Sigma}_{L,D}, \pi^* R - \sum_{i=1}^{r} n_i E_i\right) = 0,$$

where $\widetilde{\Sigma}_{L,D}$ denotes the blow up of $\Sigma_{L,D}$ at $\{z_1, \dots, z_r\}$, $\pi\colon \widetilde{\Sigma}_{L,D} \to \Sigma_{L,D}$ is the natural projection, and $E_i = \pi^{-1}(z_i)$.

Remark 4.4: (1) In principle, by imposing linear restrictions on R and $n_1, \dots, n_r$ (similar to (14)), it is possible to decrease the coefficient $(deg(D)+1)/deg(D)$.

(2) Fix $N \in \mathbb{N}$ and $\epsilon > 0$. Then Lemma 4.3 implies the following asymptotical statement: For any $d \gg 0$ and for any set of multiplicities $m_1 \leq \cdots \leq m_r < N$ satisfying

$$d^2 + O(d) \geq (1+\epsilon) \sum_{i=1}^{r} (m_i + 1)^2,$$

the following h^1-vanishing holds:

$$H^1(\mathbb{P}^2, \mathcal{I}(d)) = 0,$$

where $\mathcal{I}$ is the sheaf of ideals of the zero-dimensional scheme of generic fat points $\bigcup_{i=1}^{r} z_i^{m_i}$.

LEMMA 4.5: *Using the notation of Theorem 4.1, let $z_1, \dots, z_r \in \Sigma_{L,D}$ be points in general position. Assume that (9), (10), (11) hold. Then, for almost every section $\widetilde{\alpha} \in H^0(L, \mathcal{O}_L(M))$, there exists a curve $C \in |\mathcal{O}_{\Sigma_{L,D}}(M)|$ with the following properties:*

(i) *C has an ordinary multiple point of multiplicity $m_i = s(S_i) + 1$ at z_i for any $i = 1, \dots, r$, and no other singularities;*

(ii) *for any $i = 1, \dots, r$ the tangent directions of C at z_i are generic;*

(iii) *the curve C is connected;*

(iv) *$C \cap L$ is given by the section $\widetilde{\alpha}$.*

We postpone the proofs of the lemmas, and show first how to derive the theorem.

Proof of Theorem 4.1: We apply the patchworking procedure (Theorem 2.2) to a patchworking pattern very similar to the one presented in Example 2.1 [21]. Let us first construct the patchworking pattern.

Let $z_1, \dots, z_r \in \Sigma_{L,D}$ be points in general position. Define a flat family of surfaces $\pi\colon X = Bl_{z_1\times 0,\dots,z_r\times 0}(\Sigma_{L,D}\times\mathbb{P}^1) \to \mathbb{P}^1$ to be the blow up of $\Sigma_{L,D}\times\mathbb{P}^1$ at $z_1\times 0,\dots,z_r\times 0$. The variety X admits a natural projection $\sigma\colon X\to\Sigma_{L,D}$.

Consider a family of line bundles on $X_t = \pi^{-1}(t)$ given by the line bundle $\mathcal{L} = \sigma^*(\mathcal{O}_{\Sigma_{L,D}}(M)) \otimes \bigotimes_{i=1}^r \mathcal{O}_X(-m_iE^i)$, where E^i denotes the exceptional divisor corresponding to $z_i\times 0$. It is easy to see that the generic fiber of our family is isomorphic to $\Sigma_{L,D}$ equipped with the line bundle $\mathcal{O}_{\Sigma_{L,D}}(M)$, and the central fiber of our family is isomorphic to $\widetilde{\Sigma}_{L,D} \cup \bigcup_{i=1}^r E^i$, where $\widetilde{\Sigma}_{L,D}$ is the blow up of $\Sigma_{L,D}$ at $z_1,\dots,z_r$. The central fiber is equipped with the line bundle, whose restriction to E^i is $\mathcal{O}_{E^i}(m_i)$ and whose restriction to $\widetilde{\Sigma}_{L,D}$ is the pullback of $\mathcal{O}_{\Sigma_{L,D}}(M)$ twisted by $\mathcal{O}_{\widetilde{\Sigma}_{L,D}}(-m_iE_0^i)$ for $i=1,\dots,r$, where E_0^i is the exceptional divisor corresponding to z_i.

The last ingredient of the patchworking pattern is the section $\xi_0 \in H^0(X_0,\mathcal{L}_0)$. First, due to Lemma 4.5, there exists a curve in $|\mathcal{O}_{\Sigma_{L,D}}(M)|$ having an ordinary multiple point of multiplicity m_i at z_i for $i=1,\dots,r$, and no other singularities. Its pre-image in $\widetilde{\Sigma}_{L,D}$ is denoted by C_0^0. By Lemma 4.5, we can assume that for any i, $C_0^0\cap E^i$ is a generic set of m_i points on the line $\widetilde{\Sigma}_{L,D}\cap E^i$. Next, by Lemma 4.6 [21], we can choose good representatives $C_0^i \in |\mathcal{O}_{E^i}(m_i)|$ of the singularity types $\mathcal{S}_i$, $i=1,\dots,r$, such that $C_0^i\cap\widetilde{\Sigma}_{L,D} = C_0^0\cap E^i$. Hence the curve $\bigcup_{i=0}^r C_0^i \subset X_0$ is given by some section $\xi_0\in H^0(X_0,\mathcal{L}_0)$. Moreover, if $I_{C_0^i}\subset\mathcal{O}_{E^i}$ denotes the equisingular/equianalytic ideal sheaf of the singular point of C_0^i then $H^1(E^i, I_{C_0^i}(m_i-1)) = 0$, since $m_i = s(\mathcal{S}_i)+1$ and the set of points $C_0^i\cap\widetilde{\Sigma}_{L,D} = C_0^0\cap E^i$ is generic in $\widetilde{\Sigma}_{L,D}\cap E^i$.

Next we show that $H^1(X_0,\mathcal{I}\otimes\mathcal{L}_0)=0$, where $\mathcal{I}\subset\mathcal{O}_{X_0}$ denotes the equisingular/equianalytic ideal sheaf of the singular points of C_0. This follows from Lemma 4.3, due to Theorem 3.1 [21]. Now we can apply the weak patchworking theorem (Theorem 2.2) to derive the existence result. Namely, there exists a deformation $C_t\in|\mathcal{L}_t|$, t belonging to a small neighborhood of 0, such that C_t has exactly r singular points of types $\mathcal{S}_1,\dots,\mathcal{S}_r$ as its only singularities. If $\mathcal{I}_t$ denotes the equisingular/equianalytic ideal of the singular points of C_t, then $H^1(X_t,\mathcal{I}_t\otimes\mathcal{L}_t)=0$, by the semi-continuity of the cohomology. Hence the germ of the equisingular/equianalytic strata of C_t is smooth and has the expected dimension for small t.

To prove the irreducibility of the constructed curves we just mention that, due to Lemma 4.5, the curve C_0 is connected. Moreover, any two components of C_0 intersect transversally, and these intersection points get smoothed under the deformation. Hence the resulting curve C_t is irreducible.

Due to Lemma 4.5, for almost every section $\widetilde{\alpha} \in H^0(L, \mathcal{O}_L(M))$, we can choose C_0 in such a way that $C_0 \cap L$ is given by $\widetilde{\alpha}$. $C_t \cap L$ is a small deformation of $C_0 \cap L$, hence, for almost every section $\alpha \in H^0(L, \mathcal{O}_L(M))$, the zero set $Z(\alpha)$ is obtained as $C_t \cap L$, for an appropriate choice of t and C_0. ∎

Proof of Lemma 4.3: By the Kawamata–Viehweg vanishing theorem it suffices to show that $A = (\pi^* R - \sum_{i=1}^r n_i E_i) - K_{\widetilde{\Sigma}_{L,D}}$ is big and nef, i.e. we have to show:

(a) $A^2 > 0$, and

(b) $A.B' \geq 0$ for any irreducible curve B' in $\widetilde{\Sigma}_{L,D}$.

Note that $A = \pi^*(R - K_{\Sigma_{L,D}}) - \sum_{i=1}^r (n_i + 1) E_i$, and thus by hypothesis (13) we have

$$A^2 = (R - K_{\Sigma_{L,D}})^2 - \sum_{i=1}^r (n_i + 1)^2 > 0,$$

which gives condition (a).

For condition (b) we observe that an irreducible curve B' on $\widetilde{\Sigma}$ is either the strict transform of an irreducible curve B in Σ or one of the exceptional curves E_i. In the latter case we have

$$A.B' = A.E_i = n_i + 1 > 0.$$

We may therefore assume that $B' = \widetilde{B}$ is the strict transform of an irreducible curve B on Σ having multiplicity $mult_{z_i}(B) = k_i$ at z_i, $i = 1, \ldots, r$. Then

$$A.B' = (R - K_{\Sigma_{L,D}}).B - \sum_{i=1}^r (n_i + 1) k_i,$$

and thus condition (b) is equivalent to (b′): $(R - K_{\Sigma_{L,D}}).B \geq \sum_{i=1}^r (n_i + 1) k_i$.

Claim 4.6: *Let $B \subset \Sigma$ be an irreducible curve passing through $z_1, \ldots, z_r$ (points in general position) with multiplicities $k_1, \ldots, k_r$. Then*

$$B^2 \geq \sum_{i=1}^r k_i^2 - \min\{k_i | k_i \neq 0\}.$$

For the proof of the claim we refer to [12] Lemma 2.2. Now, using Claim 4.6, the Hodge index theorem, hypothesis (13), and the Cauchy-Schwarz inequality we get the following sequence of inequalities:

$$\begin{aligned}((R-K_{\Sigma_{L,D}}).B)^2 &\geq (R-K_{\Sigma_{L,D}})^2 \cdot B^2 \\ &\geq \left(\frac{deg(D)+1}{deg(D)}\sum_{i=1}^{r}(n_i+1)^2\right)\cdot B^2 \\ &\geq \frac{deg(D)+1}{deg(D)}\left(\sum_{i=1}^{r}(n_i+1)^2\right)\cdot\left(\sum_{i=1}^{r}k_i^2-k_{i_0}\right) \\ &=\sum_{i=1}^{r}(n_i+1)^2\cdot\sum_{i=1}^{r}k_i^2+\frac{\sum_{i=1}^{r}(n_i+1)^2\cdot(\sum_{i=1}^{r}k_i^2-(deg(D)+1)k_{i_0})}{deg(D)} \\ &\geq\left(\sum_{i=1}^{r}(n_i+1)k_i\right)^2+\frac{\sum_{i=1}^{r}(n_i+1)^2\cdot(\sum_{i=1}^{r}k_i^2-(deg(D)+1)k_{i_0})}{deg(D)},\end{aligned}$$

where $i_0 \in \{1,\dots,r\}$ is such that $k_{i_0}=\min\{k_i|k_i\neq 0\}$. Since $R-K_{\Sigma_{L,D}}$ is nef, condition (b′) is satisfied as soon as we have

$$\sum_{i=1}^{r}k_i^2 \geq (deg(D)+1)k_{i_0}, \tag{16}$$

which leaves us with the cases when $\#\{i|k_i\neq 0\}\leq deg(D)$. Moreover, we can assume that $B\neq L$ since $\underline{z}$ are in general position. $B\neq L$ is irreducible, hence

$$B.L \geq 0. \tag{17}$$

If $B=F$, then (16) is satisfied and we are done. So we can also assume that

$$B.F \geq \max\{k_i\}. \tag{18}$$

As we mentioned before $R-K_{\Sigma_{L,D}} \equiv_{num} aL+bF$, where $a=(R-K_{\Sigma_{L,D}}).F > \max\{n_i|i=1,\dots,r\}$ due to (14), and $b+aL^2=(R-K_{\Sigma_{L,D}}).L\geq 0$ which implies

$$b \geq -aL^2 = a\cdot deg(D) \geq deg(D)\cdot\max\{n_i+1\}. \tag{19}$$

Thus, due to (17), (18), (19)

$$(R-K_{\Sigma_{L,D}}).B = aL.B+bF.B > deg(D)\cdot\max\{n_i+1\}\cdot\max\{k_i\} \geq \sum_{i=1}^{r}k_i(n_i+1),$$

since $\#\{i|k_i\neq 0\}\leq deg(D)$. So condition (b′) is satisfied, and we are done. ∎

Proof of Lemma 4.5: Let $z_0 \in \Sigma_{L,D}$ be any point different from $z_1, \dots, z_r$, and let $m_0 = 0$. Define sheaves of ideals $\mathcal{J}_i$, $i = 0, \dots, r$, by

$$(\mathcal{J}_i)_z = \begin{cases} m^{m_i+1}_{\Sigma_{L,D},z_i}, & \text{if } z = z_i, \\ m^{m_j}_{\Sigma_{L,D},z_i}, & \text{if } z = z_j; j \neq i, \\ \mathcal{O}_{\Sigma_{L,D},z_j}, & \text{otherwise.} \end{cases}$$

Then define $\mathcal{J} = \sum_{i=0}^{r} \mathcal{J}_i$. By Lemma 4.3,

$$H^1(\Sigma_{L,D}, \mathcal{J} \otimes \mathcal{O}_{\Sigma_{L,D}}(M)) = H^1(\Sigma_{L,D}, \mathcal{J}_i \otimes \mathcal{O}_{\Sigma_{L,D}}(M)) = 0$$

for all $i > 0$. To show that $H^1(\Sigma_{L,D}, \mathcal{J}_0 \otimes \mathcal{O}_{\Sigma_{L,D}}(M)) = 0$ we consider the exact sequence

$$0 \to \mathcal{J} \otimes \mathcal{O}_{\Sigma_{L,D}}(M - F_0) \to \mathcal{J}_0 \otimes \mathcal{O}_{\Sigma_{L,D}}(M) \to (\mathcal{J}_0 \otimes \mathcal{O}_{\Sigma_{L,D}}(M))_{|F_0} \to 0,$$

where F_0 is the fiber of $\pi_{L,D}\colon \Sigma_{L,D} \to L$ passing through the point z_0. Since the points $z_1, \dots, z_r$ are in general position, at most one of these points belongs to F_0. Now we consider the long exact sequence,

$$H^1(\Sigma_{L,D}, \mathcal{J} \otimes \mathcal{O}(M - F_0)) \to H^1(\Sigma_{L,D}, \mathcal{J}_0 \otimes \mathcal{O}(M)) \to H^1(F_0, (\mathcal{J}_0 \otimes \mathcal{O}(M))_{|F_0}).$$

The first group is zero by Lemma 4.3, and the last group is zero by the Riemann–Roch theorem, since $M.F_0 \geq \max\{m_i | 1 \leq i \leq r\}$. Hence

$$H^1(\Sigma_{L,D}, \mathcal{J}_0 \otimes \mathcal{O}_{\Sigma_{L,D}}(M)) = 0.$$

Consider the exact sequences

$$0 \to \mathcal{J}_i \otimes \mathcal{O}_{\Sigma_{L,D}}(M) \to \mathcal{J} \otimes \mathcal{O}_{\Sigma_{L,D}}(M) \to m^{m_i}_{z_i}/m^{m_i+1}_{z_i} \to 0$$

and corresponding exact sequences of cohomology

$$H^0(\Sigma_{L,D}, \mathcal{J} \otimes \mathcal{O}_{\Sigma_{L,D}}(M)) \to m^{m_i}_{z_i}/m^{m_i+1}_{z_i} \to H^1(\Sigma_{L,D}, \mathcal{J}_i \otimes \mathcal{O}_{\Sigma_{L,D}}(M)) = 0.$$

It follows now that the generic curve in $|\mathcal{J} \otimes \mathcal{O}_{\Sigma_{L,D}}(M)|$ has an ordinary multiple point at z_i, $i = 1, \dots, r$, of multiplicity m_i with generic tangent directions at each point. It follows also that the base locus of $|\mathcal{J} \otimes \mathcal{O}_{\Sigma_{L,D}}(M)|$ is $\{z_1, \dots, z_r\}$, and hence the generic curve in this system has exactly r singular points.

Next, we have to show that the curve C is connected. We proved that the base locus of $|\mathcal{J} \otimes \mathcal{O}_{\Sigma_{L,D}}(M)|$ is finite, and thus the generic element G in this system does not contain L as a component. Since $m_i \geq 2$ for all $i = 1, \dots, r$, G contains at least one component G_1 numerically nonequivalent to kF. Then $F.G_1 > 0$

and $L.G_1 \geq 0$. Now, if G_2 is any other component of G, then $G_2 \equiv_{num} aL+bF$ for some $a \geq 0$ and $b > 0$. Hence $G_1.G_2 > 0$, which implies that G is connected.

Last, we have to show that we can choose C in such a way that $C \cap L$ is given by α. Consider the exact sequence of sheaves

$$0 \to \mathcal{J} \otimes \mathcal{O}_{\Sigma_{L,D}}(M-L) \to \mathcal{J} \otimes \mathcal{O}_{\Sigma_{L,D}}(M) \to \mathcal{O}_L(M) \to 0$$

and the corresponding long exact sequence of cohomology

$$H^0(\Sigma_{L,D}, \mathcal{J} \otimes \mathcal{O}_{\Sigma_{L,D}}(M)) \to H^0(L, \mathcal{O}_L(M)) \to H^1(\Sigma_{L,D}, \mathcal{J} \otimes \mathcal{O}_{\Sigma_{L,D}}(M-E)).$$

The last group is zero due to Lemma 4.3, and we are done. ∎

4.2 General case. Let Σ be a smooth projective algebraic surface, and let $L_1, \ldots, L_k \subset \Sigma$ be smooth curves, such that for any $1 \leq i \leq k$ and for any smooth curve $C \subset \Sigma$, the sheaf $\mathcal{O}_\Sigma(L_i)$ is ample and there exists a curve $L_i' \in |\mathcal{O}_\Sigma(L_i)|$ intersecting C transversally. In this section we will give a sufficient condition for the existence of curves with prescribed singularities in the linear systems $\mathcal{E}(\sum_{i=1}^k n_i L_i)$, where $\mathcal{E}$ is a fixed line bundle on Σ. The idea is to reduce this problem to the case of ruled surfaces by degenerating Σ to the union $\Sigma \cup \Sigma_{L_i, \mathcal{O}_{L_i}(L_i)}$, and applying geometric patchworking.

To formulate the theorem we will need the following

Notation 4.7: g_j, as usual, denotes the genus of the curve L_j. We define $a_j = n_j$, and $b_j = \deg(\mathcal{E}(\sum_{i=1}^j n_i L_i - \sum_{i=j+1}^k L_i)_{|L_j})$.

Theorem 4.8: *Let $\{S_j^i\}_{1 \leq j \leq k}^{1 \leq i \leq r(j)}$ be singularity types. Define $m_{ji} = s(S_j^i)$. Without loss of generality we can assume that $m_{j1} \geq m_{ji}$ for all i, j. Assume that there exists a smooth curve in $|\mathcal{E}(-\sum_{i=1}^k L_i)|$, and for any $1 \leq j \leq k$ the following conditions are satisfied:*

$$H^1\Big(\Sigma, \mathcal{E}\Big(-\sum_{i=1}^k L_i\Big)\Big) = 0,$$

$$(a_j+2)(2b_j - 4g_j + 4 - a_j L_j^2) \geq \frac{L_j^2+1}{L_j^2}\Big(2m_{j1} + 3 + \sum_{i=1}^{r(j)} (m_{ji}+1)^2\Big),$$

$$(a_j+2)(2b_j - 4g_j + 2 - a_j L_j^2) \geq \frac{L_j^2+1}{L_j^2} \sum_{i=1}^{r(j)} (m_{ji}+1)^2,$$

$$(a_j+1)(2b_j - 4g_j + 4 - a_j L_j^2 + L_j^2) \geq \frac{L_j^2+1}{L_j^2} \sum_{i=1}^{r(j)} (m_{ji}+1)^2,$$

$$b_j - 2g_j + 1 + L_j^2 \geq (a_j+2)L_j^2 > (m_{j1}+1)L_j^2.$$

Then, there exists a curve $C \in |\mathcal{E}(\sum_{i=1}^{k} n_i L_i)|$ *having exactly* $\sum_{i=1}^{k} r(i)$ *singular points of types* $\{\mathcal{S}_j^i\}_{1\le j\le k}^{1\le i\le r(j)}$ *as its only singularities. Moreover,* $H^1(\Sigma, I_C^{es/ea}(C)) = 0$*, and if* $k > 0$ *then* C *is irreducible.*

The following claim will be useful:

CLAIM 4.9: *Let* C *be a curve having exactly* r *singular points of types* $\mathcal{S}_1, \dots, \mathcal{S}_r$ *as its only singularities, and let* C' *be a smooth curve intersecting* C *transversally. Assume that* $H^1(\Sigma, I_C^{es/ea}(C)) = H^1(C', \mathcal{O}_{C'}(C + C')) = 0$*. Then, for generic section* $\alpha \in H^0(C', \mathcal{O}_{C'}(C+C'))$*, there exists a curve* $\widetilde{C} \in |\mathcal{O}_\Sigma(C+C')|$ *having exactly* r *singular points of types* $\mathcal{S}_1, \dots, \mathcal{S}_r$ *as its only singularities, and whose intersection with* C' *is given by* α*. Moreover,* $H^1(\Sigma, I_{\widetilde{C}}^{es/ea}(C)) = 0$.

Proof: The proof of the claim is easy; hence we present only a sketch of it and leave the details to the reader.

Consider the curve $C_0 = C \cup C'$. It has $r + C.C'$ singular points. We should mention that the equisingular/equianalytic stratum has the expected dimension at C_0 due to the given h^1-vanishing conditions. Hence by a small deformation of C_0 we can obtain a curve $\widetilde{C} \in |\mathcal{O}(C + C')|$ having exactly r singular points of types $\mathcal{S}_1, \dots, \mathcal{S}_r$ as its only singularities. Moreover, $H^1(\Sigma, I_{\widetilde{C}}^{es/ea}(C)) = 0$ due to the semicontinuity of the cohomology. Consider the exact sequence

$$H^0(\Sigma, I_{\widetilde{C}}^{es/ea}(C + C')) \to H^0(C', \mathcal{O}_{C'}(C + C')) \to H^1(\Sigma, I_{\widetilde{C}}^{es/ea}(C)) = 0.$$

It follows now that one can choose $\widetilde{C}$ in such a way that the intersection $\widetilde{C} \cap C'$ is given by a generic section α. ∎

Proof of Theorem 4.8: The proof is by induction on k. If $k = 0$ then there is nothing to prove. Assume that the result holds for $k = k_0 \geq 0$, and let us prove it for $k = k_0 + 1$.

First we construct the patchworking pattern. Let X be the blow up of $\Sigma \times \mathbb{P}^1$ along $L_k \times 0$, and let $\pi\colon X \to \mathbb{P}^1$ and $\sigma\colon X \to \Sigma$ be the natural projections. It is clear that $X_0 = \Sigma \cup E$, where $E = \Sigma_{L_k, \mathcal{O}_{L_k}(L_k)}$ is the exceptional divisor. Define line bundle $\mathcal{L}$ to be

$$\mathcal{L} = \sigma^* \mathcal{E}\left(\sum_{i=1}^{k} n_i L_i\right) \otimes \mathcal{O}_X(-n_k E).$$

It is easy to see that $\mathcal{L}_{|_\Sigma} = \mathcal{E}(\sum_{i=1}^{k-1} n_i L_i)$ and $\mathcal{L}_{|_E} \equiv_{num} n_k L + mF$, where (following the notation of the previous section) F is the homological type of the fiber in E, L is the unique irreducible curve in E having negative self-intersection, and $m = \deg(\mathcal{E}(\sum_{i=1}^k n_iL_i)_{|L_k})$.

By the induction assumption there exists a curve $C \in |\mathcal{E}(-L_k + \sum_{i=1}^{k-1} n_iL_i)|$ having exactly $\sum_{i=1}^{k-1} r(i)$ singular points of types $\{\mathcal{S}_i^j\}_{1\le i\le k-1}^{1\le j\le r(i)}$ as its only singularities. Moreover, $H^1(\Sigma, I_C^{es/ea}(C)) = 0$. Thus, due to Claim 4.9, we can construct a curve $C_1 \in |\mathcal{E}(\sum_{i=1}^{k-1} n_iL_i)|$ having exactly $\sum_{i=1}^{k-1} r(i)$ singular points of types $\{\mathcal{S}_i^j\}_{1\le i\le k-1}^{1\le j\le r(i)}$ as its only singularities, and whose intersection with L_k is given by a generic section $\alpha \in H^0(L_k, \mathcal{E}(\sum_{i=1}^{k-1} n_iL_i))$. Moreover,

$$H^1\left(\Sigma, I_{C_1}^{es/ea} \otimes \mathcal{E}\left(- L_k + \sum_{i=1}^{k-1} n_iL_i\right)\right) = 0. \tag{20}$$

Applying Theorem 4.1, we can construct a curve $C_2 \in |\mathcal{L}_{|E}|$ having exactly $r(k)$ singular points of types $\mathcal{S}_k^1, \ldots, \mathcal{S}_k^{r(k)}$ as its only singularities, and satisfying $H^1(E, I_{C_2}^{es/ea} \otimes \mathcal{L}_{|E}) = 0$. Moreover, we can assume that $C_1 \cup C_2$ is given by a section of $\mathcal{L}_0$.

Now we use Theorem 3.1 [21] and the weak patchworking theorem (Theorem 2.2) to finish the proof. ∎

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